Three Circles and Seven Tokens

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THE PROBLEM

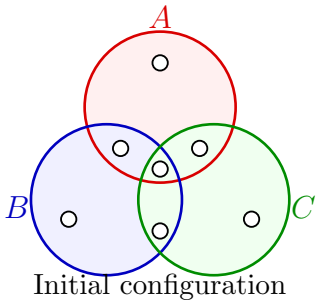
Three overlapping circles A, B, and C contain seven tokens, one in each of the seven regions of their Venn diagram. Initially, all seven tokens are white.

We are allowed to perform either of the following operations:

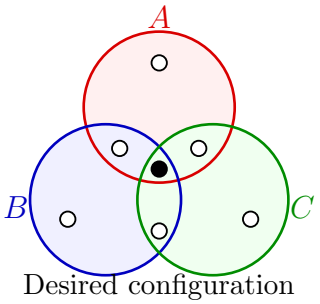
T_1: Choose one of the three circles and flip all four tokens inside it, changing white to black and black to white.

T_2: Choose one of the three circles and make all its tokens white. Equivalently, flip only those tokens in the chosen circle that are currently black.

Can we reach a configuration in which all seven tokens are white except the central token, common to all three circles, which is black?



Initial configuration

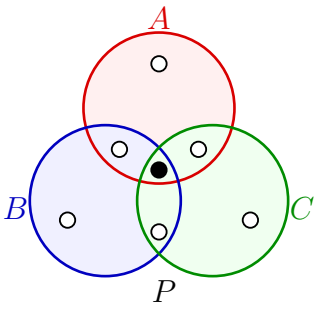


Desired configuration

SOLUTION: WORKING BACKWARDS

We want to reach a configuration in which only the central token is black. Instead of trying moves from the initial configuration, let us start from the desired one and work backwards. What could the configuration have been just before it?

Call the desired configuration P.



There are two possible kinds of moves, T_1 and T_2. Could the last move have been T_2?

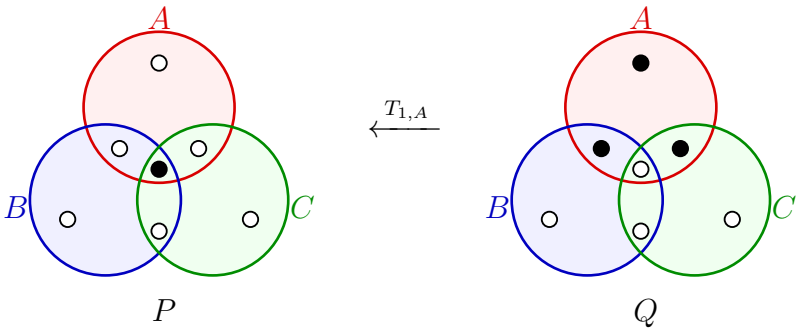
No. A T_2 move makes every token in the chosen circle white. But in P, every circle contains a black token, namely the central one. Thus the last move must have been T_1.

Now T_1 has a particularly useful property: it can undo itself. Flipping the same circle twice brings us back to where we started. In symbols,

$$T_{1,A}^2 = T_{1,B}^2 = T_{1,C}^2 = I.$$

So, if the last move was a flip of circle A, we can find the preceding configuration simply by flipping A once again. The same is true for B and C.

Let us try it. Starting from P, we could flip any of the three circles. But the three resulting pictures are the same apart from a rotation of the circle labels. There is therefore no need to draw all three. Let us flip A and call the result Q.



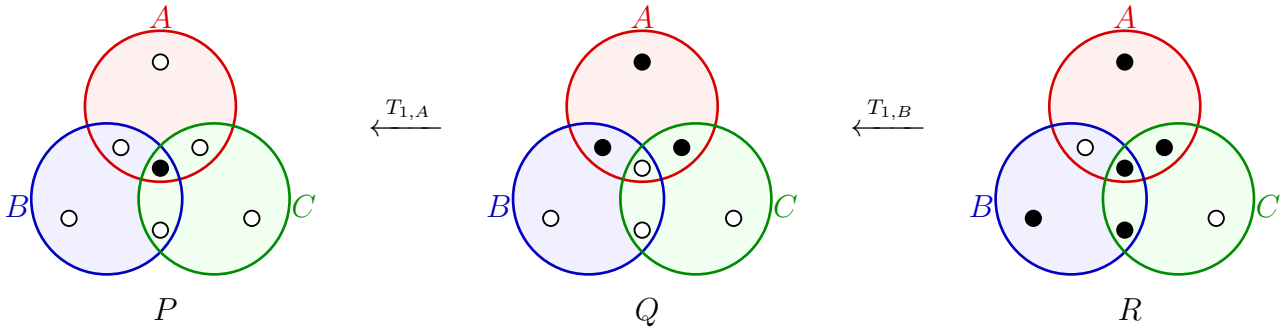
Here and below, the arrows point backwards: the configuration on the right could have preceded the one on the left.

We have found one possible preceding configuration. But can we go back another step?

Look at Q. Once again, every circle contains at least one black token. So T_2 is again impossible as the move that produced Q. We are left with T_1.

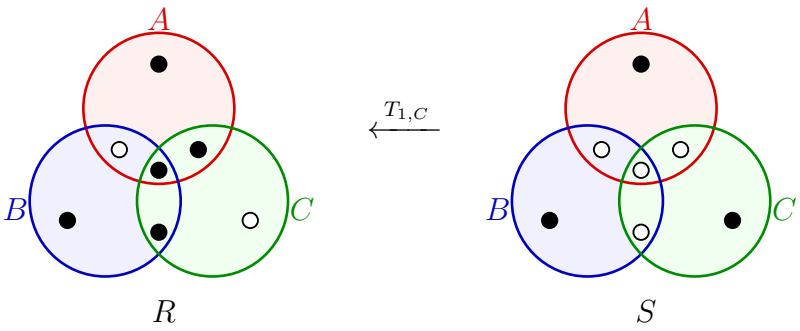
There are three choices. Flipping A takes us straight back to P. Flipping B or C produces two configurations which are rotations of one another. So there is only one genuinely new picture to consider. Call it R.

Choosing B, we get



The same question arises once more. Could R have been produced by T_2? Again, no: none of its three circles is completely white.

So we try T_1. If we flip A or B, we return to configurations of type Q, possibly with the circles interchanged. The only remaining choice is to flip C. This gives one more configuration, which we shall call S.



Now something interesting happens.

In S, all three outer tokens are black, while the remaining four are white. Every circle still contains a black token, so T_2 is impossible here as well.

What happens if we try T_1?

Whichever circle we flip, we obtain a configuration of type R. The three possibilities differ only by which circle was flipped. They are the same configuration up to a rotation.

So we have reached a point where no new kind of configuration appears.

Let us see what has happened. Working backwards from P, we found Q, then R, then S. At every stage we checked both allowed moves. The move T_2 was ruled out because every circle contained at least one black token. The move T_1 either produced the next configuration or returned us to one we had already encountered.

We can collect the possibilities in a simple diagram:

$$\boxed{P} \longleftrightarrow \boxed{Q} \longleftrightarrow \boxed{R} \longleftrightarrow \boxed{S}.$$

The arrows are reversible because T_1 is reversible. The diagram records the different shapes of configurations, with rotated copies counted as the same shape. Thus every possible T_1 move keeps us within these four types, including their rotated copies. We have checked all the possible routes, and none leads outside this family.

Also, every configuration in this family has at least one black token in each circle. Hence no T_2 move could have produced any of them.

This means that, however far we continue backwards, we remain among these same configurations. There are eight actual configurations in all: one of type P, three rotated versions of Q, three of R, and one of S.

But the initial configuration, in which all seven tokens are white, is not among them.

Therefore the all-white configuration cannot occur anywhere in the backward history of the desired configuration.

We conclude that it is impossible to reach the desired configuration from the all-white initial configuration. □

Note. This argument and solution grew out of a before-dinner blackboard discussion.